

High and Low Dimensions in The Black Hole Negative Mode

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ABSTRACT: The negative mode of the Schwarzschild black hole is central to Euclidean quantum gravity around hot flat space and for the Gregory-Laflamme black string instability. We analyze the eigenvalue as a function of space-time dimension $\lambda = \lambda(d)$ by constructing two perturbative expansions: one for large d and the other for small $d - 3$, and determining as many coefficients as we are able to compute analytically. Joining the two expansions we obtain an interpolating rational function accurate to better than 2% through the whole range of dimensions including $d = 4$.

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1. Introduction and Summary

The Schwarzschild black hole is known to possess a single off-shell negative mode¹ discovered by Gross, Perry and Yaffe (GPY) [1]. It plays a central role in Euclidean Quantum Gravity around hot flat space [1] and in the Gregory-Laflamme (GL) black string instability [2, 3, 4].

In the context of hot flat space the Euclidean black hole appears as a saddle point of the action with the same asymptotics, namely a periodic Euclidean time direction, and is interpreted to represent a non-perturbative (tunneling) decay mode through the nucleation of a real black hole. The resulting leading correction to the spacetime energy density comes from the exponential of minus the action. If this non-perturbative effect is to describe an instability this correction should be imaginary and whether this is the case is determined by the subleading correction given by the square-root of the perturbations determinant. Therefore Gross-Perry-Yaffe argue that the consistency of this picture requires an odd number of negative modes, and in particular at least one mode should exist.

The Gregory-Laflamme string instability occurs only for wavenumbers smaller than a certain critical wavenumber k_{GL} , which translates into a critical mass in the presence of a fixed compactification length. k_{GL} has a clear physical meaning and it is directly determined by the negative eigenvalue, as the latter is nothing but $(-k_{GL}^2)$.

The negative eigenvalue was computed numerically in 4d to be [1] $\lambda(4d) \simeq 0.76$ and extended to arbitrary dimensions in [5, 6]² where it was shown that the behavior of $k_{GL}(d)$ brings insight into the discovery of a critical dimension where the GL transition turns

¹This mode does not represent a physical instability of the black hole but rather of the black string constructed from it.

²The data was obtained in [5] and published in [6].

second order [5].³ The negative eigenvalue was also computed in [2] for $4 \leq d \leq 9$, in [8] for $5d$, and in [9] Harmark-Obers coordinates were used to obtain a master equation. The computation was generalized in [10] to boosted strings, in [11] to charged strings and in [12] to rotating strings. Finally related non-uniform strings in various d appeared in [13].

While an analytic determination of $\lambda(d)$ is not known it is usually useful to have some analytic insight. In this paper we attempt for analytic control by a standard strategy in physics – looking at extreme limits and perturbing around them with a small parameter. Here, taking d as a variable allows for two limits: high d and low d where for reasons to be described later at low d we are interested in $d \rightarrow 3^+$ (approaching 3 from above). Actually the large d limit of this problem was already considered in [6] but here we are able to obtain more results as discussed towards the end of this introduction.

In high dimensions gravity becomes increasingly short-ranged, and this limit was already considered as a simplified limit for the Feynman diagrams of quantum gravity [14, 15, 16]. The idea of small, continuously varying dimensions is also familiar from the technique of Dimensional Regularization in field theory [17]. Moreover, 3d gravity is known to have an interesting and somewhat degenerate limit (see [18],[19] and references therein) where gravitational waves do not exist and space-time is flat apart from deficit-angle point singularities at the location of each particle.

Recently “the optimal gauge” master eigenvalue equation for the negative mode was obtained in [20, 21]. This “optimal gauge” was generalized by proving that for perturbations of any spacetime with at most one non-homogeneous coordinate the gauge freedom can always be eliminated [22] thus avoiding the need to choose a gauge fix. The latter general theorem was already applied to a re-derivation of gravitational waves in the Schwarzschild background [23].

In this paper we utilize the new master equation to analytically study the function $\lambda(d)$ for both large and small dimensions. In section 2 we set the stage by introducing the master equation and showing that in both the high and low d limits it can be recast in terms of two different variables and analytic control of λ can be achieved through the technique of Matched Asymptotic Expansion (MAE) between the near horizon and asymptotic regions. In sections 3 and 4 we analyze the equations for high and low dimensions respectively. In section 5 we compare the analytic results of the previous two sections to new high-precision numerical data (summarized in appendix A) and propose an interpolating rational function valid over the whole range of d . The interpolation resembles the Padé approximation only here it is based on two Taylor expansions (rather than one) from both ends of the range $3 \leq d \leq \infty$.

Summary of results. The high d expansion is found to be

$$\lambda = \hat{d} - 1 + \frac{2}{\hat{d}} + \dots \equiv d - 4 + \frac{2}{d} + \dots , \quad (1.1)$$

³More insight into this critical dimension and other aspects of the problem can be gleaned from a hydrodynamical analogue [7] of the Rayleigh-Plateau instability which is responsible for the creation of drops in a faucet.

where $\hat{d} \equiv d - 3$. See a discussion of the reliability of the last term of (1.1) in the second paragraph of section 5. The low d expansion is

$$\begin{aligned}\lambda &= c_1 \epsilon + c_2 \epsilon^2 \cdots \equiv c_1 (d - 3) + c_2 (d - 3)^2 \dots \\ c_1 &\simeq 0.71515 \\ c_2 &\simeq 0.0627 ,\end{aligned}\tag{1.2}$$

where $\epsilon \equiv \hat{d} \equiv d - 3$ is small, and while the leading $\mathcal{O}(\epsilon)$ behavior was inferred analytically the constants c_1, c_2 were determined numerically.

The two expansions can be combined into the following rational approximation valid throughout the range $0 \leq \hat{d} \leq \infty$

$$\lambda(d) \simeq \hat{d} \frac{(1 - c_1) \hat{d} + c_1}{(1 - c_1) \hat{d} + 1} .\tag{1.3}$$

The approximation was built to match both series (apart for the last term of each one), and it is found to uniformly agree with the numerical data to better than 2%. In particular by approximating $c_1 \approx 0.7$ we find the following handy formula

$$\lambda \simeq \hat{d} \frac{3\hat{d} + 7}{3\hat{d} + 10} ,\tag{1.4}$$

and in 4d

$$\lambda(4d) \approx \frac{10}{13} = 0.769 ,\tag{1.5}$$

which is in good agreement with our available numerical value $\lambda(4d) = 0.76766$.

Discussion. In general the idea to study GR in various dimensions, high and low, is received with considerable resistance from both laymen and professionals. In our opinion, one of the important motivations to consider various d 's is the view that General Relativity is defined for all d , and hence d should be considered as its parameter. From this point of view it is as natural to study GR in various d 's as it is to study field theories with matter content that differs from the standard model. We view the success of the analysis presented here, including the insight into 4d results as a vindication of this “dimension as a parameter” approach. For completeness it should be mentioned that there are at least two other important reasons to study GR in various d 's: theoretical reasons from String Theory, and “phenomenological” reasons in the context of the Large Extra Dimensions and braneworld scenarios.

Comparison with [6]. In [6] the leading order high d behavior of $\lambda(d)$ was sought analytically. Here we are able to go further, getting two subleading orders as well as finding the low d behavior. Another difference is that the “optimal gauge” master equation was not available at the time of [6], but rather the transverse-traceless gauge was used, adding a spurious singularity to the equation and presumably obstructing the analysis in the near zone.

Open questions. It is plausible that our interpolating formulae (1.3,1.4) can be further refined. For instance $\lambda \simeq (d-1)(d-3)/d$ is interesting. Generalization to other perturbation problems is to be expected.⁴ Another point is that since the Matched Asymptotic Expansion analysis amounts to forgetting the asymptotic region altogether in the actual calculation, it is possible that there is a more elegant argument to demonstrate this fact.

Note added (v3). While our results are unchanged we added qualifying remarks on the assumed decoupling of the asymptotic region from the near horizon region (for both high and low d).

2. Set-up

The metric of the d -dimensional Schwarzschild black hole in standard Schwarzschild coordinates is given by

$$ds^2 = -f dt^2 + f^{-1} dr^2 + r^2 d\Omega_{d-2}^2$$

$$f(r) = 1 - \left(\frac{r_0}{r}\right)^{d-3}, \quad (2.1)$$

where r_0 , the Schwarzschild radius, is the location of the horizon and $d\Omega_{d-2}^2$ denotes the metric of the round $d-2$ sphere \mathbf{S}^{d-2} . Hereafter we shall use units such that $r_0 = 1$.

The “optimal gauge” master equation for the negative mode is [20, 21]

$$\left[-\frac{1}{r^{d-2}} \partial_r f r^{d-2} \partial_r + V(r) \right] \psi = -\lambda \psi$$

$$V(r) := -\frac{2(d-1)(d-3)^3}{r^2 (2(d-2)r^{d-3} - (d-1))^2}, \quad (2.2)$$

where ψ is a master field which is a gauge invariant combination of the metric perturbation components $h_{tt}, h_{\Omega\Omega}$ (see [20] for the full definition). Our sign conventions were chosen such that a negative eigenvalue is represented by a positive λ .

Note that the eigenvalue $\lambda = \lambda(d)$ is related to the critical Gregory-Laflamme wavenumber for the $d+1$ dimensional black string through [2]

$$\lambda = k_{GL}^2. \quad (2.3)$$

Large d is clearly an interesting limit. Numerical data suggests that in 3d λ vanishes. $d=3$ is also seen to be special from the definition of $f(r)$ (2.1). Therefore we shall attempt to analyze also $d \rightarrow 3^+$. Altogether our two limits are

- High d , namely $d \rightarrow \infty$ where the small parameter is $1/d$
- Small d , namely $d \rightarrow 3^+$ where the small parameter is $d-3$.

Accordingly we find it useful to denote

$$\hat{d} := d - 3. \quad (2.4)$$

⁴For quasi normal modes some high d behavior is already known [24].

The appearance of the term r^{d-3} in the definition of f (2.1) and in several other places in the master equation (2.2) suggests to define the variable

$$X := r^{\hat{d}} \equiv r^{d-3} . \quad (2.5)$$

The master equation turns out to have a nice representation in the X variable

$$\begin{aligned} [-\partial_X X(X-1) \partial_X + V_X(X)] \psi &= -\frac{r^2 \lambda}{\hat{d}^2} \psi \\ V(X) &:= -\frac{2(\hat{d}+2)\hat{d}}{\left(2(\hat{d}+1)X - (\hat{d}+2)\right)^2} , \end{aligned} \quad (2.6)$$

gotten by multiplying (2.2) by r^2/\hat{d}^2 , and here r should be considered to be a function of X .

It will also be convenient to define

$$\hat{r} := r - 1 \quad \hat{X} := X - 1 , \quad (2.7)$$

such that the horizon lies at $0 = \hat{r} = \hat{X}$.

In both high and low d limits we can define two regions (or zones) on the r, X axis namely, the near horizon region and the asymptotic region, and moreover their overlap will increase in size as the limit is taken. According to the method of matched asymptotic expansion, one solves for $\lambda = \lambda(d)$ as follows. One solves for $\psi = \psi(r; \lambda)$ separately in each region each time with only a single boundary condition - regularity at the boundary. The second boundary condition comes from matching in the overlap zone. Normally when propagating a wavefunction from two boundaries one requires ψ'/ψ to be continuous at the matching point. When we have a matching region, rather than a point, ψ can be characterized as a linear combination of two basic solutions, namely we denote the general solution inside the overlap region (which is usually simpler) $\psi = A(\lambda) \psi_1(r; \lambda) + B(\lambda) \psi_2(r; \lambda)$ where A, B are two arbitrary coefficients. The matching condition becomes

$$\left. \frac{A}{B} \right|_{\text{near}} (\lambda) = \left. \frac{A}{B} \right|_{\text{asympt}} (\lambda) , \quad (2.8)$$

where A, B from each region need to be read from the asymptotics of $\psi_{\text{region}}(r; \lambda)$ away from the boundary. This is the condition that fixes the eigenvalue λ .

Actually in the two limits under consideration, matching will not be necessary: we shall start the next two sections by demonstrating that in both cases it is enough to analyze the near horizon region and to replace the matching boundary conditions by a second regularity condition in the far side of the near horizon region zone. This means that both wave functions are localized in the near horizon region. ⁵

⁵Note that there are examples where this simplification does not occur. For example, in the 1d Quantum Mechanics problem of finding a bound state for a shallow potential, $[-\partial_x^2 + \epsilon V(x)]\psi = E(\epsilon)\psi$ the wave function of the bound state extends much beyond the size of the potential.

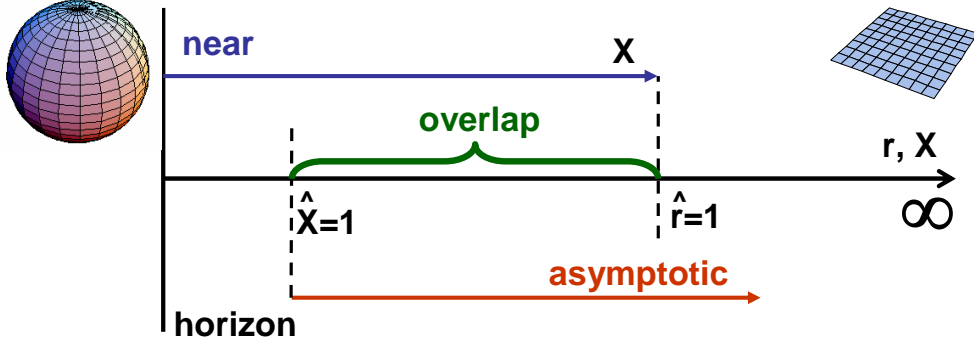


Figure 1: The regions along the radial direction at high dimension.

3. High d

For high d we have $\hat{r} \ll \hat{X}$ and hence $\hat{X} = 1$ occurs much before $\hat{r} = 1$ on the r, X axis (see figure 1). Accordingly we can define two regions

- Near horizon $\hat{r} \ll 1$
- Asymptotic $\hat{X} \gg 1$

and the two regions have an overlap which increases in size as the large d limit is taken, in agreement with the general requirements of the method of matched asymptotic expansion.

Asymptotic region and boundary conditions for the near horizon region. In the asymptotic region $f \simeq 1$ and $V \simeq 0$ and hence the background is simply flat d -dimensional space-time. In this region the master equation becomes

$$-\frac{1}{r^{d-2}} \partial_r r^{d-2} \partial_r \psi = -\lambda \psi . \quad (3.1)$$

λ can be rescaled out of this equation after defining the dimensionless coordinate $\tilde{r} := \sqrt{\lambda} r$, and the solution satisfying regularity at infinity is

$$\psi = \frac{K_{\hat{d}/2}(\tilde{r})}{\tilde{r}^{\hat{d}/2}} , \quad (3.2)$$

where $K_{\hat{d}/2}$ is the modified Bessel function of the second kind of order $\hat{d}/2$.

Overlap region. In the overlap region both $\hat{r} \ll 1$ and $\hat{X} \gg 1$, the λ term in the equation becomes negligible and hence both r and X equations (2.2,2.6) become

$$-\frac{1}{\tilde{r}^{d-2}} \partial_{\tilde{r}} \tilde{r}^{d-2} \partial_{\tilde{r}} \psi = 0 = -\partial_X X^2 \partial_X \psi ,$$

whose general solution is

$$\begin{aligned} \psi &= A + B \frac{1}{X} \\ &= A + B \frac{1}{r^{\hat{d}}} . \end{aligned} \quad (3.3)$$

By examining the near horizon behavior of the Bessel function (3.2) we convinced ourselves that to leading order

$$A = 0 . \quad (3.4)$$

Below we shall adopt a working assumption that this boundary condition (which implies that the near horizon region does not receive “communications” from the asymptotic region) is correct to all order which we compute. We view the good agreement of the analytic and numerical results as a confirmation of this assumption (see the second paragraph of section 5).

The eigenvalue problem. In the near horizon region X is a good variable ⁶ and we rewrite the X master equation (2.6) in a form convenient for the high d limit

$$[-\partial_X X(X-1)\partial_X + V(X)]\psi = -r^2(X)\tilde{\lambda}\psi , \quad (3.5)$$

where the rescaled eigenvalue is

$$\tilde{\lambda} := \frac{\lambda}{\hat{d}^2} , \quad (3.6)$$

while the potential and $r^2(X)$ are given by

$$V(X) = -\frac{2\left(1 + 2/\hat{d}\right)}{\left(2X - 1 + 2(X-1)/\hat{d}\right)^2} \quad (3.7)$$

$$r^2 = \exp\left(\frac{2}{\hat{d}}\log X\right) . \quad (3.8)$$

Zeroth order. The zeroth order equation in the near horizon region as $\hat{d} \rightarrow \infty$ is

$$L_h \psi = -\tilde{\lambda} \psi , \quad (3.9)$$

where the horizon zeroth order differential operator is given by

$$L_h := -\partial_X X(X-1)\partial_X - \frac{2}{(2X-1)^2} . \quad (3.10)$$

The boundary conditions for this equation are regularity both at the horizon $\hat{X} = 0$ and asymptotically at $\hat{X} \rightarrow \infty$. More specifically, near the horizon the two independent solutions behave as *const*, $\log(\hat{X})$ and we require that the coefficient of the $\log(\hat{X})$ piece vanishes. Asymptotically the two solutions behave as *const*, $1/\hat{X}$ and we require that the constant vanishes.

The zeroth order equation (3.9) has a *marginally bound state*, namely a normalizable solution with $\tilde{\lambda} = 0$ given simply by

$$\psi_0 = \frac{1}{2X-1} . \quad (3.11)$$

⁶An alternative change of variables which is of certain convenience is $\xi := -\tanh(\log(f)/2) = 1/(2X-1)$.

Indeed this solution satisfies the boundary conditions both at the horizon and at infinity and it has finite norm

$$\langle \psi_0 | \psi_0 \rangle = \frac{1}{2} \quad (3.12)$$

under the inner product which makes L_h Hermitian. This inner product is nothing but the standard L^2 norm

$$\langle \phi | \psi \rangle = \int_1^\infty \phi^*(X) \psi(X) dX . \quad (3.13)$$

Since ψ_0 has no nodes we also conclude that it is the ground state, and hence $\tilde{\lambda} = 0$ at $\mathcal{O}(\hat{d}^0)$.

We proceed by the standard perturbation theory for an eigenvalue problem. At each order k we find an equation of the form

$$L_h \psi_k = -\tilde{\lambda}_k \psi_0 + Src_k , \quad (3.14)$$

where $\tilde{\lambda}_k$, ψ_k are the coefficients in the $1/\hat{d}$ expansion of the eigenvalue and eigenfunction, respectively

$$\tilde{\lambda} = \sum_{k=0}^{\infty} \tilde{\lambda}_k / \hat{d}^k \quad \psi = \sum_{k=0}^{\infty} \psi_k / \hat{d}^k . \quad (3.15)$$

Src_k is the source term at the k 'th order which comes from the $1/\hat{d}$ expansion of (2.6) and involves $\tilde{\lambda}$ and ψ from lower orders. More specifically one needs to expand in (2.6) the near horizon potential, $V(\hat{X})$, and $r^2 = X^{2/\hat{d}} = \exp(2 \log(X)/\hat{d})$.

Equation (3.14) is solved first by taking the inner product with ψ_0 to find ⁷

$$\tilde{\lambda}_k = \frac{\langle \psi_0 | Src_k \rangle}{\langle \psi_0 | \psi_0 \rangle} . \quad (3.16)$$

Next ψ_k is solved from the inhomogeneous differential equation (3.14) while imposing regularity at the boundaries. This task is facilitated by the existence of a simple solution (3.11) to the homogenous equation and by the use of the Wronskian. At each order there is a residual freedom to add to ψ_k any multiple of ψ_0 which amounts to a \hat{d} -dependent normalization factor for ψ . While it could be fixed by requiring ψ_k to be orthogonal to ψ_0 , it is not necessary to do so, and we use this freedom to simplify the expression for ψ_k when possible.

First order. The first order correction to the potential is

$$V_1 = -\frac{4}{(2X-1)^3} \quad (3.17)$$

and the first source term is

$$Src_1 = -V_1 \psi_0 . \quad (3.18)$$

⁷Note that the boundary condition (3.4) was used to derive this, and that a different one could have required to add a boundary term.

Hence by (3.16) the first correction to $\tilde{\lambda}$ is

$$\tilde{\lambda}_1 = -\frac{\langle \psi_0 | V_1 | \psi_0 \rangle}{\langle \psi_0 | \psi_0 \rangle} = 1 . \quad (3.19)$$

Solving (3.14) for ψ_1 we find

$$\psi_1 = \frac{1}{(2X-1)^2} - \frac{\log X}{2X-1} . \quad (3.20)$$

Second order. The second order correction to the potential is

$$V_2 = +\frac{8(X-1)(X+1)}{(2X-1)^4} \quad (3.21)$$

and the second order source term is

$$Src_2 = [-V_2 - 2 \log X \tilde{\lambda}_1] \psi_0 + [-V_1 - \tilde{\lambda}_1] \psi_1 , \quad (3.22)$$

where the $2 \log X$ term comes from expanding r^2 .

By (3.16)

$$\tilde{\lambda}_2 = \frac{\langle \psi_0 | Src_2 \rangle}{\langle \psi_0 | \psi_0 \rangle} = -1 . \quad (3.23)$$

Solving (3.14) for ψ_2 we find

$$\psi_2 = \frac{1}{(2X-1)^3} - \frac{\log X + 1}{(2X-1)^2} + \frac{Li_2(1-X)}{2X-1} , \quad (3.24)$$

where $Li_2(x) \equiv \sum_{k=1}^{\infty} \frac{x^k}{k^2}$ is the second poly-logarithm function.

Third order. At this order we can compute analytically the correction to the eigenvalue

$$\tilde{\lambda}_3 = 2 , \quad (3.25)$$

where we used

$$V_3 = -\frac{16(X-1)^2(2X+1)}{(2X-1)^5} . \quad (3.26)$$

Here we stop the perturbative expansion as we did not obtain analytically the fourth order correction to the eigenvalue.

Summarizing the high d section, we found that assuming the boundary condition (3.4) the eigenvalue is given by

$$\lambda = \hat{d} - 1 + \frac{2}{\hat{d}} + \dots \equiv d - 4 + \frac{2}{d} + \dots \quad (3.27)$$

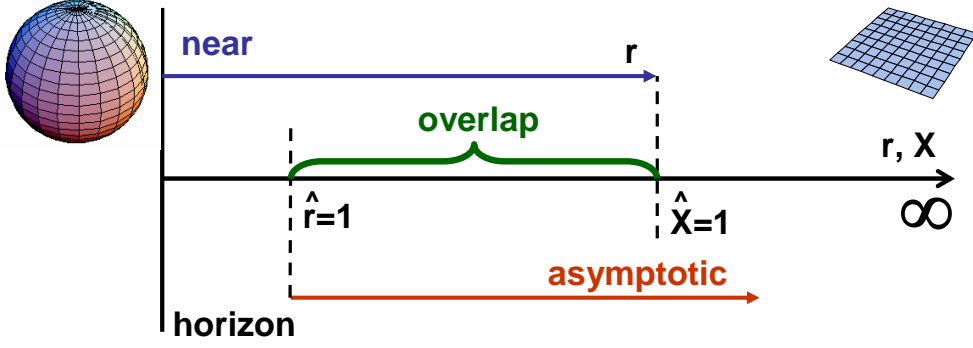


Figure 2: The regions along the radial direction at low dimension.

4. Low d

In the low d limit the analysis runs in parallel with the previous high d analysis, but some features differ. We shall see that the matching problem still reduces to the near horizon region, but here we can determine analytically only the order of leading behavior and we need to resort to numerical analysis even in order to determine the corresponding leading coefficient.

Since we are taking the limit $d - 3 \rightarrow 0$ it will be convenient to use another notation for \hat{d}

$$\epsilon := \hat{d} \equiv d - 3 . \quad (4.1)$$

Unlike the large d limit, here $\hat{X} \ll \hat{r}$, and hence $\hat{r} = 1$ occurs much before $\hat{X} = 1$ on the r, X axis (see figure 2). Accordingly we can define two regions

- Near horizon $\hat{X} \ll 1$
- Asymptotic $\hat{r} \gg 1$

and again the two regions have an overlap which increases in size as the low d limit is taken, allowing to employ the method of matched asymptotic expansion.

Asymptotic region and boundary conditions for the near horizon region. In the asymptotic region \hat{X} is a good coordinate, although $r \gg X$ would continue to be useful. In this region

$$-V(r) = \frac{2\epsilon^3 (2 + \epsilon)}{r^2 \left(2(1 + \epsilon) \hat{X} + \epsilon \right)^2} \leq \frac{\epsilon^3}{r^2 \hat{X}^2} \ll \lambda \quad (4.2)$$

and hence the potential can be neglected (in the last inequality it should be borne in mind that $\lambda = \mathcal{O}(\epsilon)$ as we shall see later in this section).

On the other hand the function $f(r) = 1 - r^{-\epsilon} = \hat{X}/(1 + \hat{X})$ varies in this region: for large \hat{X} we have $f \simeq 1$ while for small \hat{X} we have $f \simeq \hat{X} \simeq \epsilon \log(r)$. Yet, throughout the asymptotic region the leading expression for the solutions is

$$\psi = A \exp(\sqrt{\lambda} r) + B \exp(-\sqrt{\lambda} r) . \quad (4.3)$$

Therefore regularity at infinity requires choosing $A = 0$. Again we shall make a working assumption that this boundary condition is correct to all orders which we compute and we view the good agreement with the numerical results as a confirmation.

The eigenvalue problem. In the near horizon region r is a good coordinate. The master equation reads (2.2)

$$\begin{aligned} & \left[-\frac{1}{r^{1+\epsilon}} \partial_r f(r) r^{1+\epsilon} \partial_r + V(r) \right] \psi = -\lambda \psi \\ f &= 1 - \frac{1}{r^\epsilon} \\ V &= -\frac{2\epsilon^3 (2 + \epsilon)}{r^2 (2(1 + \epsilon) (r^\epsilon - 1) + \epsilon)^2} . \end{aligned} \quad (4.4)$$

For fixed r we have $r^\epsilon - 1 = \exp(\epsilon \log(r)) - 1 = \epsilon \log(r) + \dots$. Hence for small ϵ we may approximate f by

$$f = \epsilon \log(r) + \dots \quad (4.5)$$

In order to have an expression for $V(r)$ in this limit we expand the following expression from its denominator

$$Dn_V := 2(1 + \epsilon) (r^\epsilon - 1) + \epsilon = \epsilon(2 \log(r) + 1) + \dots \quad (4.6)$$

Altogether the zeroth order eigenvalue problem is

$$\left[-\frac{1}{r} \partial_r r \log(r) \partial_r - \frac{4}{r^2 (2 \log(r) + 1)^2} \right] \psi = -\bar{\lambda} \psi , \quad (4.7)$$

where in this section the rescaled eigenvalue is defined by

$$\bar{\lambda} := \lambda / \epsilon , \quad (4.8)$$

which is different from the high d definition. The scalar product which makes this operator Hermitian is

$$\langle \phi | \psi \rangle = \int_0^\infty \phi^*(r) \psi(r) r dr . \quad (4.9)$$

Curiously for $\bar{\lambda} = 0$ there is again an exact solution

$$\psi = \frac{1 - 2 \log(r)}{1 + 2 \log(r)} . \quad (4.10)$$

It is regular at both $r = 1$ and $r = \infty$ but non-normalizable. For us it is important that it has a single node thereby indicating the existence of a unique bound state.⁸ Unlike high d here the marginally bound state is found not to become bound for $\epsilon > 0$ – which is fortunate, since we are expecting only a single bound state.

⁸Here normalizability is not necessary. Imagine increasing λ while keeping regularity at the horizon. During this process $\psi(r)$ increases and the node approaches ∞ . The λ where the node becomes infinite is the sought for ground state.

So far we have shown the existence of a single bound state whose eigenvalue scales as $\lambda = c_1 \epsilon + \dots$ where c_1 is some positive constant. In particular

$$\lambda(3d) = 0 . \quad (4.11)$$

We were not able to solve for c_1 analytically. But a numerical analysis of (4.7) yields $c_1 = .71515$.

One can proceed to higher orders in perturbation theory $\lambda = c_1 \epsilon + c_2 \epsilon^2 + \dots$. Clearly this needs to be done numerically as even the zeroth order eigenvalue and eigenfunction are known only numerically. Doing so we obtained $c_2 = 0.0627$.

In summary the low d expansion is

$$\begin{aligned} \lambda &= c_1 (d - 3) + c_2 (d - 3)^2 + \dots \\ c_1 &= 0.71515 \\ c_2 &= 0.0627 . \end{aligned} \quad (4.12)$$

5. Numerical data and interpolations

We need high-precision numerical data for $\lambda(d)$ in order to confirm our perturbative results and in order to evaluate the success of various interpolating functions based on these expansions. To that end we computed $\lambda(d)$ to 5-6 digits of precision by fairly standard numerical procedures and the results are collected in a table in appendix A.

We first confirm that the expansions which we derived (1.1,1.2) are consistent with the data. The method is straightforward: to test an order k Taylor expansion around r_0 assuming order $k - 1$ was confirmed already, we deduct the partial sum up to order $k - 1$ from the data, divide by the small parameter to power k and seek to identify a limit as $r \rightarrow r_0$ (extrapolations may be used to increase the accuracy of the limit). Using this method we confirm the first two coefficients in (1.1,1.2), thereby giving us added confidence in the validity of our boundary conditions. The agreement with the $2/d$ term in the high d expansion is less clear. The remainder function which estimates the coefficient of the $1/d$ term is indeed very close to 2 for $12 \lesssim d \lesssim 21$ but then for higher d it increases and gets as high as 2.5 and 3. We interpret this increase to be due to either the diminishing numerical accuracy for high d caused by the multiplication by d or to a correction to the boundary condition (3.4) at this order.

Next we would like to exploit our control over the function at the two boundaries in order to construct an interpolating function which would be a reasonable approximation across the whole range $3 \leq d < \infty$. We shall require the function to coincide with the derived Taylor expansion on both sides up to a prescribed order. A popular and simple kind of interpolating function is provided by the class of rational functions. Note that while the Padé approximation also employs rational functions, it is usually fitted to a Taylor expansion at a single point and is therefore an *extrapolation* which is much less reliable than the current *interpolation*.

It is observed that λ/\hat{d} is a slowly varying function. A simple possibility for a rational interpolating function is

$$\frac{\lambda}{\hat{d}} \simeq \frac{a_0 + a_1/\hat{d}}{b_0 + b_1/\hat{d}}. \quad (5.1)$$

This equation has 3 unknowns (an overall multiplication of all constants does not change the function) and thus can be made to fit 2 of the coefficients at high d and the single c_1 coefficient of low d . The resulting rational function is

$$\Lambda(d) = \hat{d} \frac{(1 - c_1) \hat{d} + c_1}{(1 - c_1) \hat{d} + 1}. \quad (5.2)$$

The approximation $\Lambda(d)$ agrees quite well with the data for $\lambda(d)$, as can be seen from figures 3,4.

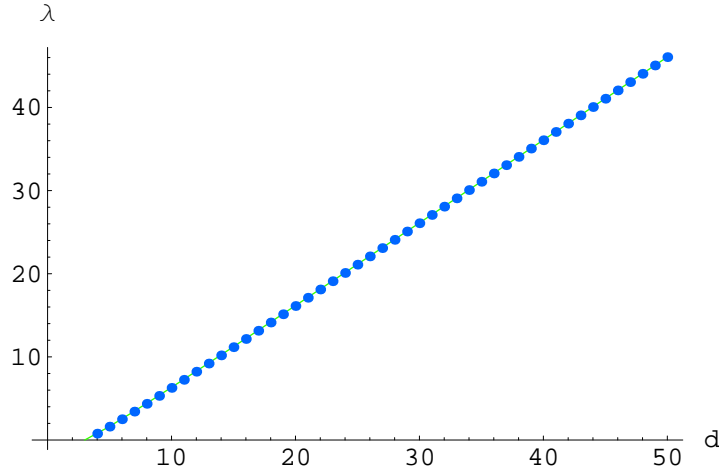


Figure 3: The negative eigenvalue λ as a function of d the space-time dimension. The blue dots represent the numerical data (see appendix A) while the solid green line represents the interpolation (5.2) with $c_1 = 0.71515$.

We also attempted interpolation with rational functions of higher order, and we found that not only does their approximation error not improve relative to (5.2) but it is actually even larger.

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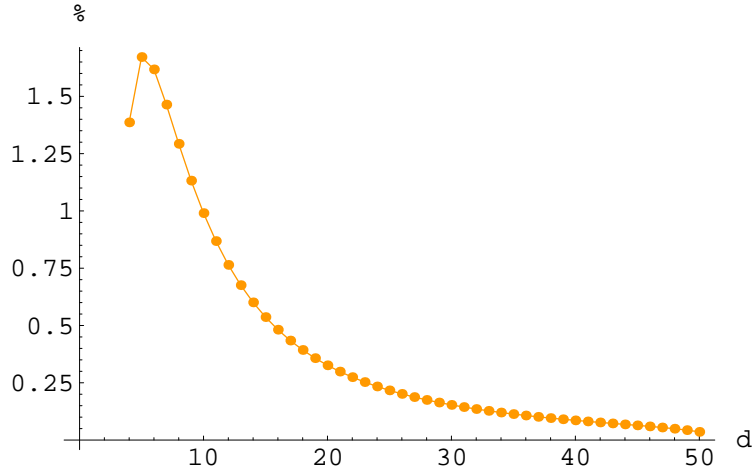


Figure 4: The error of interpolation in percent as a function of d (the data is represented by the heavy dots and the solid curve is intended merely to guide the eye). It is seen that the maximal error is less than 1.7% and for $d \geq 10$ it is even less than 1%.

A. High precision numerical values for $\lambda(d)$

In this section we present in table 1 high precision numerical results for $\lambda(d)$ which is defined by the eigenvalue problem (2.2). For more details on the method of calculation see the Mathematica notebook [26].

d	4	5			
λ	0.76766	1.61015			
d	6	7	8	9	10
λ	2.49879	3.41739	4.35618	5.30901	6.27189
d	11	12	13	14	15
λ	7.24210	8.21779	9.19766	10.1808	11.1664
d	16	17	18	19	20
λ	12.1541	13.1434	14.1341	15.1259	16.1186
d	22	24	26	28	30
λ	18.1062	20.0962	22.0879	24.0810	26.0750
d	32	34	36	38	40
λ	28.0700	30.0656	32.0618	34.0585	36.0557
d	42	44	46	48	50
λ	38.0533	40.0515	42.0504	44.0505	46.0530

Table 1: Numerically computed high-precision negative eigenvalues λ of (2.2) (in units of r_0^{-2}).

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